A Robust AFPTAS for Online Bin Packing with Polynomial Migration

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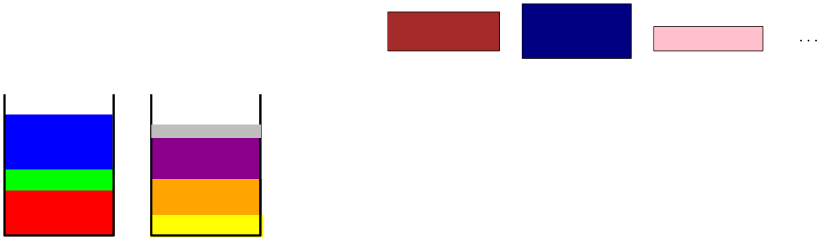
Online Bin Packing



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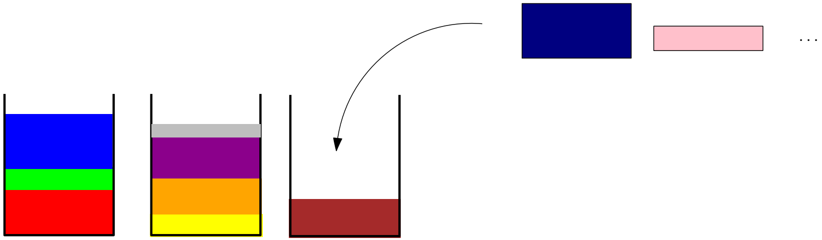

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Given an instance $I_t = \{i_1, \ldots i_t\}$ of items for each time $t \in \mathbb{N}$ and a function $s : I = \bigcup_t I_t \to [0,1]$.

Find for each $t \in \mathbb{N}$ an assignment $B_t : \{i_1, \ldots, i_t\} \to \mathbb{N}^+$ such that $\sum_{i:B_t(i)=j} s(i) \leq 1$ for all j.

Goal: Minimize the number of bins $\max_i \{B_t(i)\}$ for each time t.

Competitive Ratio of Online Bin Packing

Best known algorithm: Ratio of 1.58889 (S.S. Seiden, 2002)

Best known lower bound: Ratio of 1.54037 (J. Balogh, B. Jozsef, and G. Galambos, 2010)

Offline Bin Packing

APTAS:
Approximation guarantee: $(1+\epsilon)OPT+1$
Running time: $\text{poly}(n)+f(1/\epsilon)$ (W. Fernandez de la Vega and G.S. Lueker, 1981)

AFPTAS:
Approximation guarantee: $(1+\epsilon)OPT+O(1/\epsilon^2)$
Running time: $\text{poly}(n, 1/\epsilon)$ (N. Karmarkar and R.M. Karp, 1982)

Repacking

Online Bin Packing with Repacking

- Ratio 1.33 repacking 7 items (G. Gambosi et al., 2000)
- Ratio 1.25 repacking $\mathcal{O}(\log t)$ "shifting moves" (Z. Ivkovic and E.L. Lloyd, 1998)
- Ratio $1 + \epsilon$ repacking amortized $\mathcal{O}(\log t)$ "shifting moves" (Z. Ivkovic and E.L. Lloyd, 1997)

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Online Scheduling: Achieving ratio < 1.4659 requires repacking of $\Theta(t)$ jobs (S. Albers and M. Hellwig, 2012)

Migration

Migration Factor between B_t and B_{t+1}

$$\frac{1}{s(i_{t+1})} \sum_{j \leq t : B_t(i_j) \neq B_{t+1}(i_j)} s(i_j)$$

An algorithm is *robust* if the migration factor is bounded by a function $f(1/\epsilon)$. (P. Sanders, N. Sivadasan and M. Skutella, 2004).

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  3. **YES** and **NO** Online Machine Covering (M. Skutella and J. Verschae 2010)

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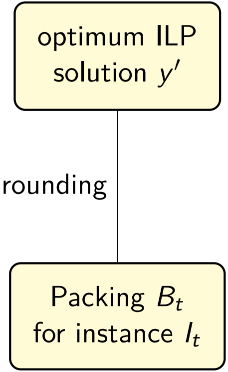
- Can we achieve polynomial migration and polynomial running time?

Our Result for Online Bin Packing:

Approximation scheme with running time poly$(t, 1/\epsilon)$ and migration factor poly$(1/\epsilon)$.

Overview Robust Algorithms



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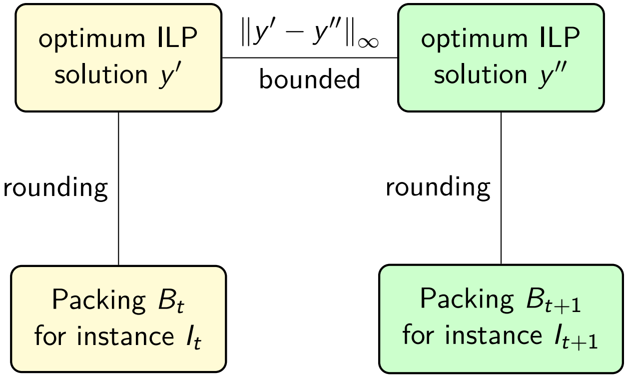

Sensitivity Analysis

Goal

Let y' be an optimum ILP solution of $\min\{\|x\|_1 \,|Ax \geq b', x \geq 0\}$. Find an optimum ILP solution y'' of $\min\{\|x\|_1 \,|Ax \geq b'', x \geq 0\}$ such that $\|y'' - y'\|_\infty$ is small.

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Theorem (Cook et al., 1986): There exists an optimum ILP solution y'' with $\|y'' - y'\|_\infty \leq n\Delta(\|b'' - b'\|_\infty + 2)$.

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Problem: The number of variables n and the largest subdeterminant Δ can only bounded by an exponential term in $1/\epsilon$.

Theorem

Consider the LP $\min \{\|x\|_1 \mid Ax \geq b, x \geq 0\}$ *with* $A \in \mathbb{R}_{\geq 0}^{m \times n}$ *and let* x' *be an approximate fractional solution with* $\|x'\|_1 \leq (1+\delta)OPT$ *for* $\delta > 0$ *and* $\|x'\|_1 \geq \alpha(\frac{1}{\delta} + 1)$.

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Then there exists a solution x'' *with* $\|x''\|_1 \leq (1+\delta)OPT - \alpha$ *and* $\|x' - x''\|_1 \leq 2\alpha\,(1/\delta + 1)$.

Prove the feasibility of the following linear program:

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \\ \sum x_i &\leq (1+\delta)OPT - \alpha \\ x &\geq x' - \alpha(1/\delta + 1)\frac{x'}{\|x'\|_1} \\ x &\leq x' + \alpha(1/\delta + 1)\frac{x^{OPT}}{\|x'\|_1} \end{aligned}$$

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A feasible solution is $x'' = (1 - \frac{\alpha(1/\delta+1)}{\|x'\|_1})x' + \frac{\alpha(1/\delta+1)}{\|x'\|_1}x^{OPT}$.

Algorithm

Let x' *be a LP solution with* $\|x'\| \leq (1+\delta)OPT$ *and* $\|x'\| \geq \alpha(1/\delta + 1)$.

- *Set* $x^{fix} := x' - \frac{\alpha(1/\delta+1)}{\|x'\|_1} x'$ *and* $b^{var} := b - A(x^{fix})$
- *Solve the LP* $\hat{x} = \min\{\|x\|_1 \,|Ax \geq b^{var}, x \geq 0\}$
- *Generate a new solution* $x'' = x^{fix} + \hat{x}$

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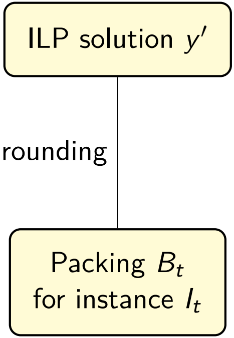
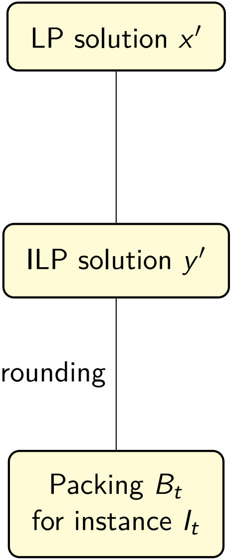
The algorithm returns a feasible LP solution x'' with $\|x''\|_1 \leq (1+\delta)OPT - \alpha$ and distance $\|x'' - x'\|_1 \leq 2\alpha(1/\delta + 1)$.

Improve Packing:

Let B_t be a packing of instance I_t with $\max_i B_t(i) \leq (1+\delta)OPT$.

Find a packing B'_t with $\max_i B'_t(i) \leq (1+\delta)OPT - 1$ such that migration factor between B_t and B'_t is small.


ILP solution y'
rounding
Packing B_t for instance I_t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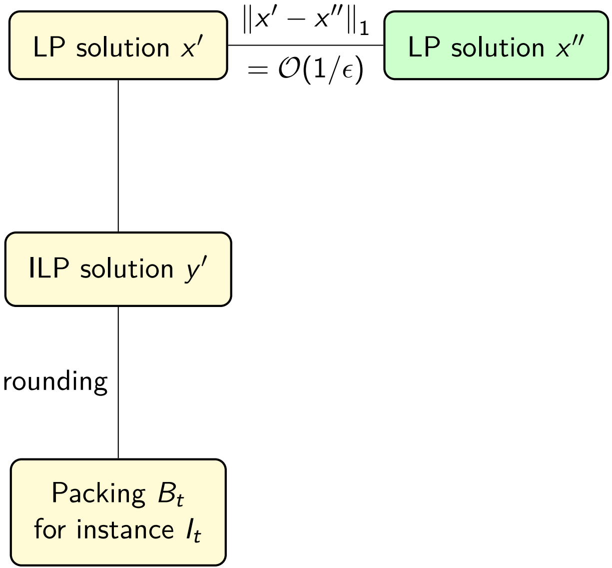
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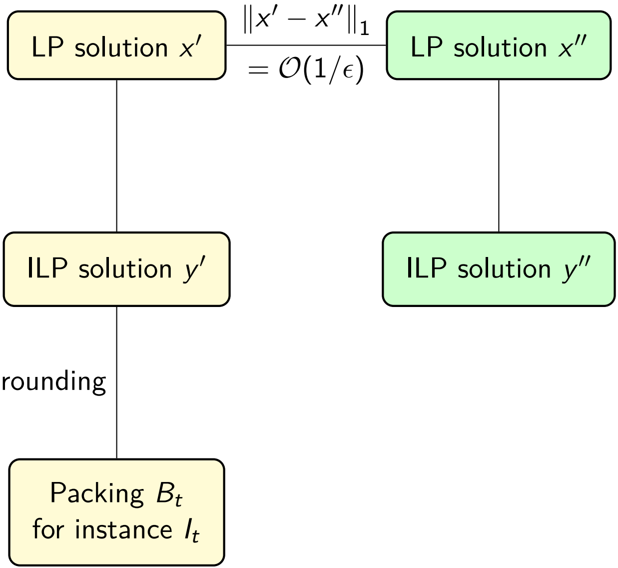
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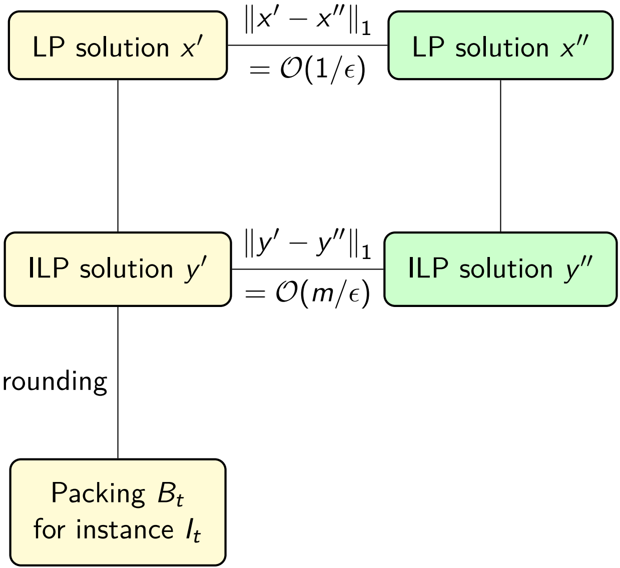
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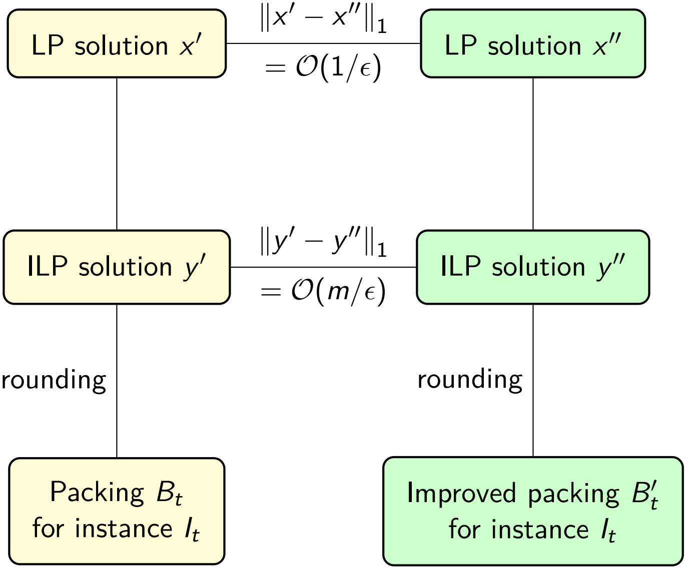
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- Keep the number of non-zero components small
- Avoid calculation of optimum LP solutions
- Dynamic rounding technique

Main Result

We obtain a fully robust AFPTAS for the online bin packing problem with migration factor $\mathcal{O}(1/\epsilon^4)$ and running time $\mathcal{O}(M(1/\epsilon^2)1/\epsilon^4 + \epsilon t + 1/\epsilon^2 \log(\epsilon^2 t))$.

Open Questions:

- Smaller migration factor and running time
- Lower bounds for migration factor
- Dynamic bin packing (allow departing of items)
- Use LP-techniques for other online problems

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Thank you!