Scheduling Malleable Task Trees

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ENS Lyon

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Outline

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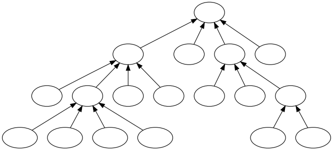
Introduction

Motivation

- Solving sparse linear systems $\longrightarrow$ sparse matrix factorizations
  $\longrightarrow$ task trees to be scheduled
- Processing power available: homogeneous parallel platform
- Need to schedule task trees using tree and task parallelism



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- **task tree:** structure defining precedence order, a node cannot begin before its children are completed
- **tree parallelism:** possibility to execute simultaneously several tasks
- **task parallelism:** possibility to allocate several processors to a task

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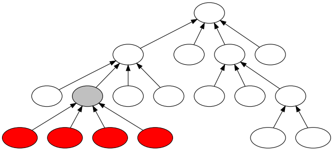

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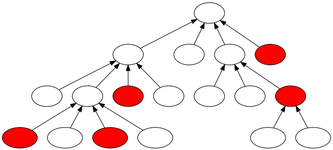

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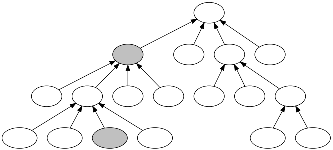

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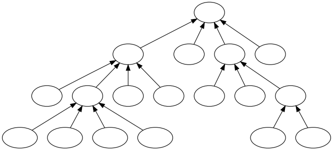

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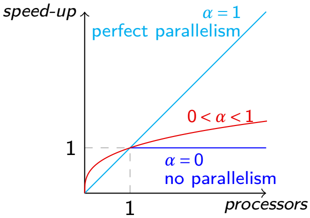
Model and notations

Parameters of the problem

- Need for a model of realist (imperfect) task parallelism: **Malleable tasks** [Le04]
- **Tree graph** G (previous slide)
- **Processor profile**: step function $p(t)$, available number of processors at time t

Speedup f (= sequential time / parallel time)

- $f(p) = p^{\alpha}$ for $0 < \alpha < 1$, $p \in \mathbb{R}^+$ (non-integer processor shares: time-sharing techniques)
  Advocated for matrix computations [PM96,BG07]
- Processing time of task T_i on p processors: L_i / p^{α}



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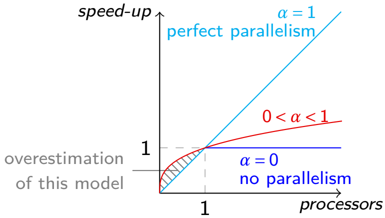


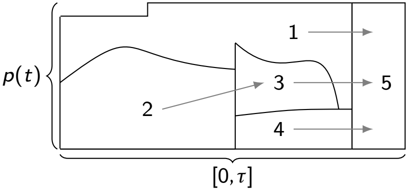
Definition of schedules

Structure of schedules

- Schedule $\mathscr{S}$: piecewise continuous functions $\{t \mapsto p_i(t)\}$ defined on $[0, \tau]$
- τ: makespan of $\mathscr{S}$ (supposed tight: not all $p_i(\tau - \varepsilon)$ are null)
- Ratio of work up to time t: $w_i(t) = \int_0^t p_i(x)^\alpha \, \mathrm{d}x / L_i$

Validity conditions of a schedule

- Does not use more than $p(t)$ processors at any time t: $\sum_i p_i(t) \le p(t)$
- Completes all the tasks: $\forall i,\ w_i(\tau) = 1$
- Respects the precedence order: $\forall i, \forall t \in [0, \tau],\ w_i(t) > 0 \implies \forall j \in Children(T_i), w_j(t) = 1$



Generalization of trees

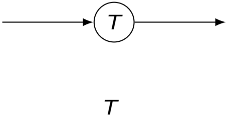
Our objective: study trees
Next two sections: study a more general structure

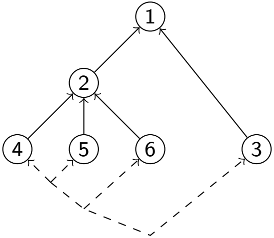
Series Parallel graphs

Recursively defined by being either:

- a single task
- a parallel composition of two SP graphs
- a series composition of two SP graphs

A tree can be extended to a SP graph.





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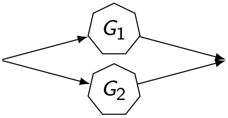
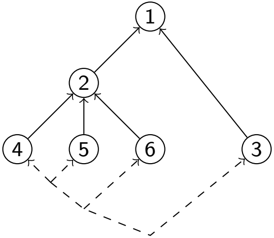


$G_1 \parallel G_2$



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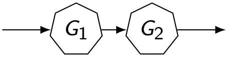

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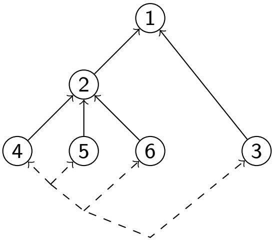


Outline

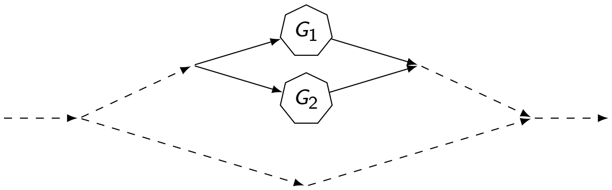
Statement of the problem

Context

- Objects of interest: miminum-makespan schedules of a SP graph G
- [PM96] proved the theorem below using *heavy* optimal control theory
- Our objective: reprove it using pure-scheduling arguments

Theorem (Prasanna & Musicus)

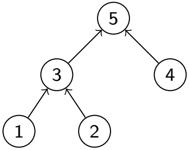
*Optimal schedules respect the **Processor Flow Conservation property**:*
the ratio of processors given to each branch of any parallel node is constant.

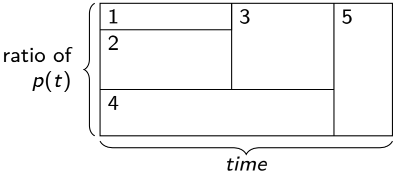


Consequences of the theorem

Corollary

- *Each task: alloted a constant ratio, independent of $p(t)$*
  its children terminate simultaneously
- *Each graph G is equivalent to the task of length $\mathscr{L}_G$ recursively defined by:*
  - $\mathscr{L}_{T_i} = L_i$
  - $\mathscr{L}_{G_1\,;\,G_2} = \mathscr{L}_{G_1} + \mathscr{L}_{G_2}$
  - $\mathscr{L}_{G_1 \parallel G_2} = \left(\mathscr{L}_{G_1}^{1/\alpha} + \mathscr{L}_{G_2}^{1/\alpha}\right)^{\alpha}$
- *The (unique) optimal schedule $\mathscr{S}_{\mathrm{PM}}$ can be computed in polynomial time.*





A tree G (particular SP graph) and the shape of its optimal schedule under any $p(t)$

First step of the proof: $p_i(t)$'s are step functions

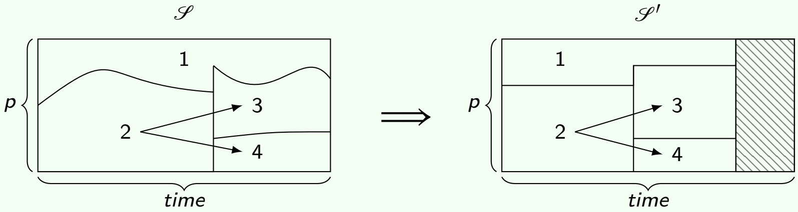
A **clean interval** of a schedule $\mathscr{S}$: a time interval during which no task terminates.

Lemma

If $p(t) = p$, optimal schedules have constant $p_i(t)$'s on its clean intervals.

Proof.

- Consider $\mathscr{S}$ with $p_j(t)$ not constant on a clean $\Delta \longrightarrow \mathscr{S}'$ with smaller makespan
- Uses strict concavity of f: replace $p_i(t)$'s by their mean
- Get the inequality: $W_j^\Delta(\mathscr{S}) = \int_\Delta p_j(t)^\alpha dt \quad < \quad \int_\Delta \left(\frac{1}{\Delta}\int_\Delta p_j(t)dt\right)^\alpha dx$



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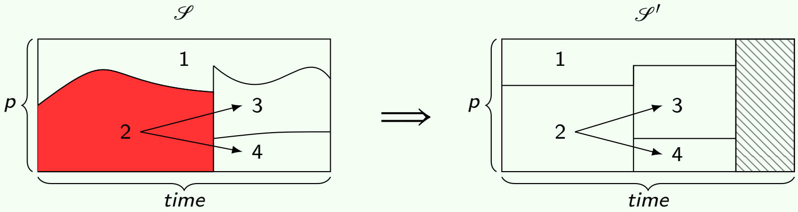

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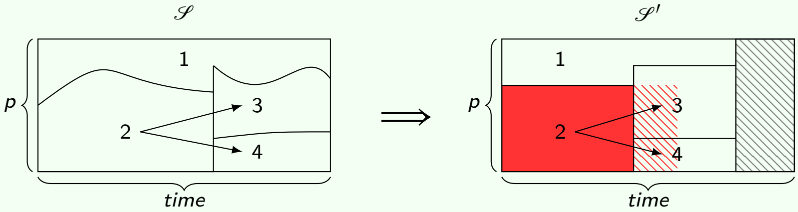

For any G: let $r_i(t) = p_i(t)/p(t)$ be the fraction of processors allocated to T_i.

Lemma

For G being $T_1 \parallel T_2$, in optimal schedules: $r_1(t) = L_1^{1/\alpha} \Big/ \mathscr{L}_{1\parallel 2}^{1/\alpha}$.

Proof. *(Note that $p(t)$ is not necessarily constant)*

- Suppose $\mathscr{S}$ optimal with $r_1(t)$ not constant $\longrightarrow$ $\mathscr{S}'$ with a smaller makespan
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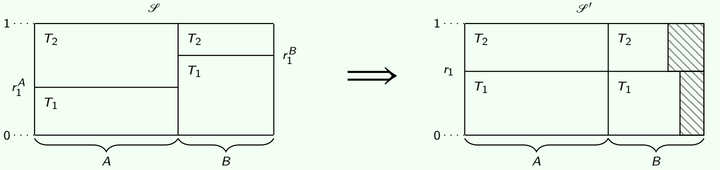

Details: with $Ap_A^\alpha = Bp_B^\alpha$ and $2r_1 = r_1^A + r_1^B$,

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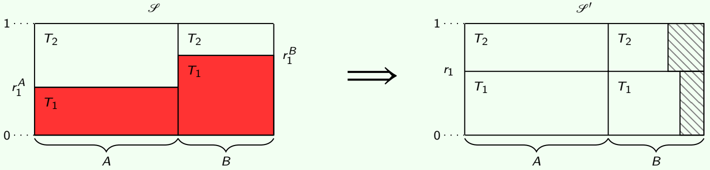

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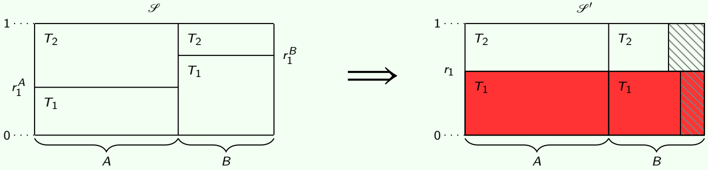

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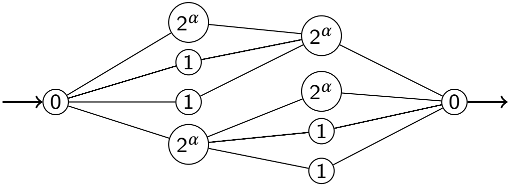
End of the proof of the theorem

Few steps remaining to prove the theorem:

- $T_1 \parallel T_2$ under any $p(t)$ $\iff$ $T_{1\parallel 2}$ of length $\mathscr{L}_{1\parallel 2}$ under any $p(t)$
- $T_1\,;T_2$ under any $p(t)$ $\iff$ $T_{1;2}$ of length $\mathscr{L}_{1;2}$ under any $p(t)$
- Proof by induction on the structure of G

- $p(t) = 6$



Example of computed schedule

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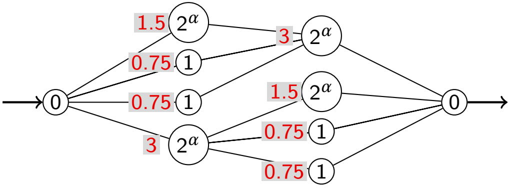
- $T_1 ; T_2$ under any $p(t)$ $\iff$ $T_{1;2}$ of length $\mathscr{L}_{1;2}$ under any $p(t)$
- Proof by induction on the structure of G

- $p(t) = 6$
- $M = \left(\frac{2}{3}\right)^\alpha + \left(\frac{4}{3}\right)^\alpha$



Example of computed schedule

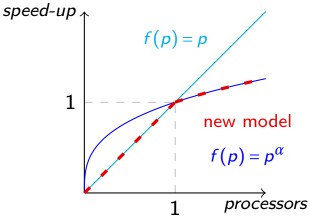
Outline

Refinement of the model

Motivation: the previous model overestimates the speedup for $p < 1$

Modification of the speedup function

- $p \geq 1$: $f(p) = p^\alpha$
- $p \leq 1$: $f(p) = p$



Consequences

The previous theorem does not hold.
We cannot compute the optimal schedule.
Restriction: assume $p(t) = p$ in the following.

Consequence of the refinement

Definition (PM allocation)

The allocation $\mathscr{S}_{\text{PM}}$ computed by the formulas of previous section.

Theorem

The PM allocation is not a constant ratio approximation at α fixed.

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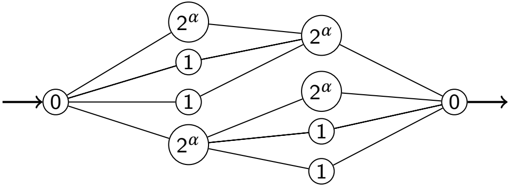

Example of graph where the PM allocation is not optimal

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The PM allocation is not a constant ratio approximation at α fixed.

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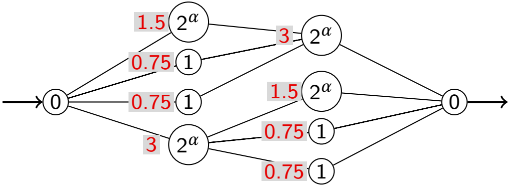
- PM schedule, optimal with previous model
- $M_1 = \left(\frac{2}{3}\right)^{\alpha} + \frac{4}{3}$



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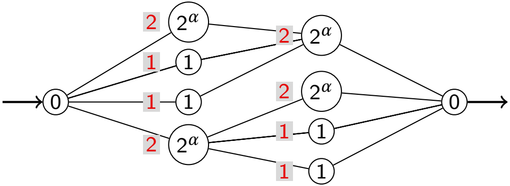
- $p(t) = 6$
- Better schedule
- $M_2 = 2 < M_1$



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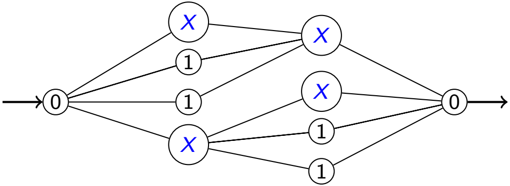

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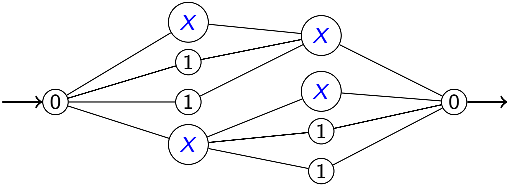


Example of graph where the PM allocation is not optimal

Need to extend the study to more general allocations...

PFC allocations

Need for a more general structure, close to an optimal solution, and simple to study

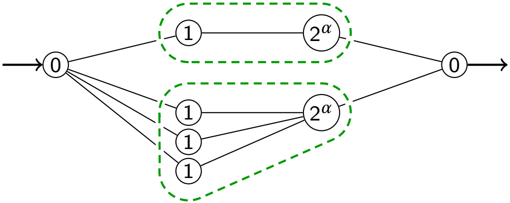
Definition (PFC allocation)

An allocation that allocates a constant ratio to each subgraph at every parallel node.

Theorem

The (unique) best PFC allocation is not always the optimal schedule, even in the restriction to pseudo-trees.

- $p(t) = 4$



Example of pseudo-tree graph illustrating the theorem

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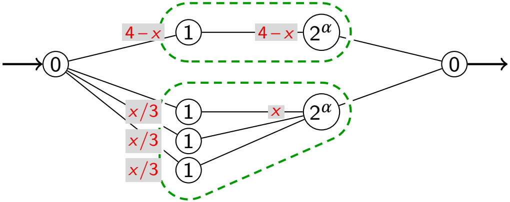
- $p(t) = 4$
- PFC schedules
- $M_1(x, \alpha) > 2$



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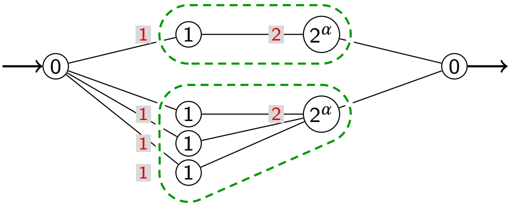
- Better schedule
- $M_2 = 2 < M_1(x, \alpha)$



Example of pseudo-tree graph illustrating the theorem

PFC allocations

Need for a more general structure, close to an optimal solution, and simple to study

Definition (PFC allocation)

An allocation that allocates a constant ratio to each subgraph at every parallel node.

Theorem

The (unique) best PFC allocation is not always the optimal schedule, even in the restriction to pseudo-trees.

Remark (best PFC allocation seen as an approximation)

Approximation ratio $< p^{1-\alpha}$.
For $\alpha = 1/2$*: approximation ratio* > 1.09 $\longrightarrow$ *the exact ratio is unknown.*

Remark

Possibility to check if a PFC allocation is the best one (existence of idle times)...
...but not to compute it.

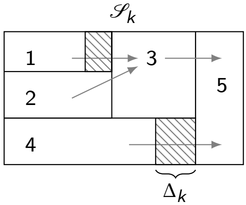
Heuristic towards the computation of the best PFC allocation

Principle of the heuristic

- In the PM schedule: makespan of tasks with $p_i < 1$ is underestimated
- Artificially increase their processor need
- Goal: find $\overline{L_i}$ from L_i such that $L_i / p_i = \overline{L_i} / p_i^\alpha \quad \longrightarrow \quad \overline{L_i} := L_i \cdot p_i^{\alpha-1} > L_i$

Iterative algorithm

1. `Initialisation:` $G_0 \leftarrow G$
2. `Repeat step` k `until (hoped) convergence:`
   - `compute the PM schedule` $\mathscr{S}_k$ `of` G_k
   - `modify the` L_i`'s with` $p_i < 1$ `to create` G_{k+1}



Elements towards its correctness for $\alpha > 1/2$

- Convergence is proved on $T_1 \parallel T_2$
- Observations on random/selected graphs:
  - For any graph G the heuristic converges
  - Both Δ_{2k} and Δ_{2k+1} decrease and converge to 0

Δ_k: *largest idle time of* $\mathscr{S}_k$

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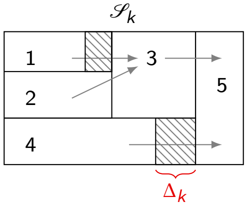

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Outline

Description of the model

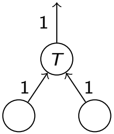
Memory: constraint on parallel platforms for direct sparse matrix factorization methods

Objective

Complexity results on schedules trying to minimize both makespan and memory peak

Assumptions on the instance of the problem

- G is a **tree**, $f(p) = p^{\alpha}$ and $p(t)$ is constant
- Tasks have **output files**
- While executing a task, input and output files must be allocated
- In our proofs: file sizes are equal to 1 and lengths to 0 or 1



Lemma (backbone of the following theorems)

Regardless general memory constraints, under the hypotheses:

- G: *$k \times n$ independent tasks of length 1*
- $p(t) = k \times p$
- *processing more than k tasks simultaneously is forbidden*

*Minimum makespan is reached **iff** successive batches of k tasks are scheduled.*

Description of the model

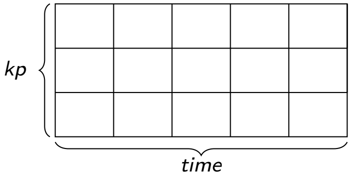


Illustration of the optimal schedule, for $k = 3$ and $n = 5$

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NP-completeness of the bi-objective problem

The BiObjectiveParallelTreeScheduling problem

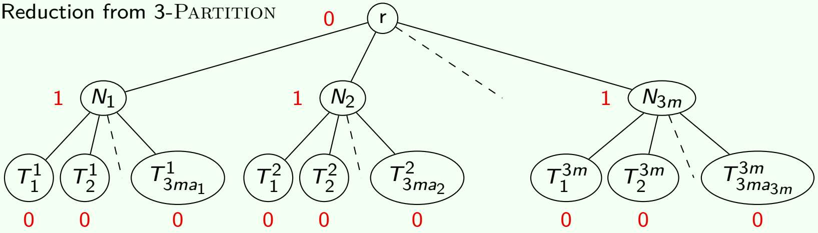
Given a valid instance: is there a schedule respecting $\{makespan < B_{C_{max}}\}$ and $\{memory\ peak < B_{mem}\}$?

Theorem

The BiObjectiveParallelTreeScheduling problem is NP-Complete.

Proof.

Reduction from 3-PARTITION



Inapproximation results

Theorem (unbounded number of processors)

There is no algorithm that is both a β-approximation for the makespan and a γ-approximation for the memory peak.

Theorem (fixed number of processors)

There is no algorithm with $\beta(p)$ and $\gamma(p)$ verifying:

$$\gamma(p)\beta(p)^{1-\alpha} \leq \left(\frac{p}{\log p + 1}\right)^{1-\alpha}$$

Remark (Comparison with previous bounds)

Without task parallelism [MSV13]:

$$\gamma(p)\beta(p) > \frac{2p}{\lceil \log p \rceil + 2}$$

Here, assuming $\alpha = 0$:

$$\gamma(p)\beta(p) > \frac{p}{\log p + 1}$$

Outline

Conclusion

Model $f(p) = p^\alpha$ for all p

- Results of [PM96] are proved using pure-scheduling arguments

Model $f(p) = p$ for $p < 1$

- PM schedules are not λ-approximations, PFC schedules are not optimal
- A heuristic probably converges towards the PFC optimal schedule for $\alpha > 1/2$

Memory-aware model

- Deciding if there exists a schedule that respects a makespan and a memory constraint is NP-complete
- There is a lower bound over the approximation ratios, coherent with the state-of-the-art bound without task parallelism